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Cyclic One-Factorization of the Complete Graph

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It is shown that the complete graph K_n has a cyclic 1-factorization if and only if n is even and $n \neq 2^t$, $t \geq 3$.

1. INTRODUCTION

We consider the complete graph of order n, K_n as the set E of all 2-subsets of the set $V = \{0, 1, 2, \ldots, n-1\}$. A 1-factorization of K_n is a partition of E into $n-1$ one-factors, so that each 1-factor is a partition of V. The existence of a single 1-factor requires that n be even, and it has been known since at least 1859 [7] (cf. also [2], [5]) that 1-factorizations of K_n exist for all even n. Henceforth, we assume n to be even.

An automorphism of a 1-factorization is a permutation of V which maps 1-factors onto 1-factors. A 1-factorization is *cyclic* if it admits an n-cycle as an automorphism. The 1-factorizations constructed in [7] admit an $(n-1)$ cycle, and this is true of almost all of the known constructions for 1-factorizations. In this note we show that cyclic 1-factorizations exist if and only if n is even and $n \neq 2^t$, $t \geq 3$. We also enumerate cyclic 1-factorizations of K_n for $n \leq 16$.

Without loss of generality we assume that a cyclic 1-factorization admits the automorphism α where $\alpha(x) = x + 1 (\text{mod } n)$.

The set of edges E is partitioned into orbits by $\langle \alpha \rangle$. For each $x \in Z_n$ define $|x|$ by

$$|x| = \begin{cases} x, & \text{if } 0 \leq x \leq n/2 \\ n - x, & \text{otherwise.} \end{cases}$$

Two edges $[a, b]$ and $[c, d]$ are in the same edge-orbit if and only if $|a-b| = |c-d|$; accordingly we denote the edge orbits as $E_1, E_2, \ldots, E_{n/2}$, where

$$E_i = \{[a, b]: |a-b| = i\}, \qquad 1 \leq i \leq n/2.$$

Note that $|E_i| = n$ if $i < n/2$ and $|E_{n/2}| = n/2$. The short edge orbit $E_{n/2}$ is itself a 1-factor, stabilized by $\langle \alpha \rangle$. Any odd edge orbit E_{2j+1} with $2j+1 \neq n/2$ can be split into two 1-factors, which are preserved by $\langle \alpha \rangle$, as follows:

$$F_1 = \{[2i, 2i+2j+1]: 0 \leq i < n/2\}$$

$$F_2 = \{[2i+1, 2i+2j+2]: 0 \leq i < n/2\}.$$

The difficulty in constructing a cyclic 1-factorization thus lies in partitioning the long even edge orbits $E_2 \cup E_4 \cup \cdots$ together with $E_{n/2}$ and/or some of the E_{2j+1} into 1-factors which are preserved by $\langle \alpha \rangle$.

We establish the impossibility of such a partition when $n = 2^t$, $t \geq 3$ in the following section.

2. THE STRUCTURE OF CYCLIC 1-FACTORIZATIONS

Let $F_1, F_2, \ldots, F_{n-1}$ be a cyclic 1-factorization of K_n. The action of $\langle \alpha \rangle$ partitions the 1-factors into orbits. The number of 1-factors in an orbit must be a divisor of n, say m, and the subgroup $\langle \alpha^m \rangle$ must stabilize each 1-factor in such an orbit. Furthermore, a 1-factor orbit is composed of a union of edge-orbits. It follows immediately that a 1-factor

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orbit of length 1 must be $E_{n/2}$ and a 1-factor orbit of length 2 must be a partition F_1, F_2 of a long odd edge orbit E_{2j+1} as given above. Hence the long even edge orbits must be assigned to 1-factor orbits of length 3 or more.

LEMMA 2.1. *A 1-factor orbit of even length cannot contain the short edge orbit $E_{n/2}$.*

PROOF. The total number of edges in 1-factor orbit of length $2m$ is mn; however, $E_{n/2}$ contains $n/2$ edges and all other E_i contain n edges.

LEMMA 2.2. *A 1-factor orbit of even length contains an even number of even edge orbits.*

PROOF. Let F be a 1-factor in a 1-factor orbit of length $2m$. Since F is stabilized by $\langle\alpha^{2m}\rangle$, F must contain m edges from distinct edge orbits, and these edges, considered modulo $2m$, must constitute a 1-factor of Z_{2m}. The number of even vertices covered by odd edges equals the number of odd vertices covered by odd edges. Hence, the same property must hold for the even edges. But even edges cover 2 vertices of the same parity, hence the number of even edges per 1-factor must be even.

COROLLARY 2.3. *If $n=2^t$, $t\geq 3$ then no cyclic 1-factorization of K_n exists.*

PROOF. The number of long even edge orbits is $2^{t-2}-1$, which is odd for $t\geq 3$. Since the length of a 1-factor orbit must be a divisor of n, all 1-factor orbits must be of even length except possibly one of length 1. The result then follows from Lemmas 2.1 and 2.2.

3. CONSTRUCTION OF CYCLIC 1-FACTORIZATIONS

THEOREM 3.1. *A cyclic 1-factorization of K_n exists if and only if n is even and $n\neq 2^t$, $t\geq 3$.*

PROOF. Necessity follows from Corollary 2.3 above. To establish sufficiency we construct cyclic 1-factorizations as follows. Let $n=2^t m$, with m odd.

Case 1 $t=1$: Let F_0 be the $\langle\alpha^m\rangle$-orbit of the following edges: $[0, m], [1, -1]$, $[2, -2], \ldots, [[m/2], -[m/2]]$. F_0 is a 1-factor since

$$\left\{0, \pm 1, \pm 2, \ldots, \pm\left[\frac{m}{2}\right]\right\} \equiv \{0, 1, 2, \ldots, m-1\} (\text{mod } m).$$

Let $F_0, F_1, \ldots, F_{m-1}$ be the set of $\langle\alpha\rangle$ images of F_0. This set covers all the even edge orbits and the short orbit E_m. All the remaining edge orbits are odd edge orbits and so may each be covered by two 1-factors stabilized by $\langle\alpha^2\rangle$.

Case 2 $t=2$: Let F_0 be the $\langle\alpha^{2m}\rangle$ orbit of the following edges: $[0, m]$, $[1, -1], [2, -2], \ldots, [m-1, 1-m]$, and let $F_0, F_1, \ldots, F_{2m-1}$ be the $\langle\alpha\rangle$ images of F_0. This set covers all the even edge orbits and the odd edge orbit E_m. The short orbit E_{2m} provides one more 1-factor, and the remaining edge orbits are odd.

Case 3 $t\geq 3$, $m\geq 3$: Let F_0 be the $\langle\alpha^{2^{t-1}m}\rangle$ orbit of the following edges: $[0, 2^{t-1}m-1]$, $[2^{t-2}m-1, 2^{t-1}m-2]$, $[1, 2^{t-1}m-3]$, $[2, 2^{t-1}m-4], \ldots, [2^{t-2}m-2, 2^{t-2}m]$. Let F_0, $F_1, \ldots, F_{2^{t-1}m-1}$ be the $\langle\alpha\rangle$ images of F_0. This set covers all the long even edge orbits with the exception of the edges of length $2^{t-1}m-2$ and also covers the two odd orbits of lengths $2^{t-1}m-1$ and $2^{t-2}m-1$.

Let G_0 be the $\langle\alpha^m\rangle$ orbit of the edges $[0, 2^{t-1}m]$, $[1, 2^{t-1}m-1]$, $[2, m-2]$, $[3, m-3]$, $\ldots$, $[[m/2], [m/2]+1]$ (empty if $m=3$). Let $G_0, G_1, \ldots, G_{m-1}$ be the $\langle\alpha\rangle$ images of G_0. This set covers the even edge orbit of length $2^{t-1}m-2$, the short edge orbit and the odd edge orbits of length $1, 3, 5, \ldots, m-4$. Since $m-4<2^{t-2}m-1$, these odd orbits have not been covered by the F_is.

The remaining edge orbits are all odd.

COROLLARY 3.2. *A transitive 1-factorization of K_n (i.e., one with an automorphism group transitive on the vertices) exists for all even n.*

PROOF. For $n=2^t$ there exists a 1-factorization whose automorphism group contains the elementary Abelian 2-group (see e.g. [2]). The remaining cases follow from Theorem 3.1.

TABLE 1
Cyclic 1-factorizations of K_{12} and K_{14}

Solution	$\langle\alpha^m\rangle$ orbits of edges				m	*T*-vectors			
						12	8+4	6+6	4+4+4
Cyclic 1-factorization for n = 12									
1	0, 1	2, 4	3, 11		6	35	0	8	12
	0, 3	1, 6			4				
	0, 6				1				
2	0, 3	1, 11	2, 10		6	34	0	8	13
	0, 1	2, 7			4				
	0, 6				1				
3	0, 1	2, 4	3, 11		6	21	12	5	7
	0, 5	1, 7			3				
	0, 3				2				
4	0, 3	1, 11	2, 10		6	25	6	15	9
	0, 1	2, 8			3				
	0, 5				2				
5	0, 1	2, 4	3, 11		6	18	6	14	17
	0, 3				2				
	0, 5				2				
	0, 6				1				
6	0, 3	1, 11	2, 10		6	20	0	20	15
	0, 1				2				
	0, 5				2				
	0, 6				1				
						T-vectors			
						14	10+4	8+6	6+4+4
Cyclic 1-factorization for n = 14									
1	0, 7	1, 5	2, 10	4, 6	7	15	0	42	21
	0, 1				2				
	0, 3				2				
	0, 5				2				
2	0, 7	1,13	2, 12	3, 11	7	78	0	0	0
	0, 1				2				
	0, 3				2				
	0, 5				2				

4. Cyclic 1-Factorizations of K_n for Small n

Nonisomorphic 1-factorizations of K_n have been enumerated exactly for $n \geqslant 10$ (see [3]). In addition, work has been done on enumerating 1-factorizations having some additional properties. E.g. 1-factorizations admitting an $(n-1)$-cycle on vertices (cyclic on 1-factors) have been enumerated for $n \leqslant 16$ [4], perfect 1-factorizations for $n \leqslant 12$ [6], and those with both of the above properties for $n \leqslant 20$ [1].

We enumerated (by hand) all cyclic 1-factorizations of K_n for $n \leqslant 16$. The number $C(n)$ of cyclic 1-factorizations of K_n is as follows:

n	4	6	8	10	12	14	16
$C(n)$	1	1	0	1	6	2	0

In Table 1 we list all cyclic 1-factorizations of K_{12} and K_{14} (those for $n \leqslant 10$ are given by Theorem 3.1), together with their T-vectors. Roughly speaking, the T-vectors count the number of types of unions of two 1-factors (see [6]; cf. also [2]). In the above range, the T-vectors distinguish completely between nonisomorphic cyclic 1-factorizations.

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